# Math 132: Differential Topology

## § Integration on manifolds

Recall from vector calculus that, if $f: V \to U$ is a diffeomorphism of open sets in $\mathbb{R}^m$ and $a$ is an integrable function on $U$, then

$$\int_U a \, dx_1 \cdots dx_m = \int_V (a \circ f) \, |\det(df)| \, dy_1 \cdots dy_m .$$

Notice the similarity with the pull back formula:

$$f^*(a \, dx_1 \wedge \cdots \wedge dx_m) = (a \circ f) \det(df) \, dy_1 \wedge \cdots \wedge dy_m .$$

It follows that:

Thm (Change of variables in $\mathbb{R}^m$)

If $f: V \to U$ is an orientation-preserving diffeomorphism of open sets in $\mathbb{R}^m$ (or $\mathbb{H}^m$) and $\omega$ is an integrable $m$-form on $U$, then

$$\int_U \omega = \int_V f^* \omega .$$

If $f$ reverses the orientation, then $\int_U \omega = - \int_V f^* \omega$.

2/

**Def** Let $M$ be an oriented $m$-manifold and let $\omega$ be a smooth, compactly supported $m$-form on $M$.

Define the integral of $\omega$ over $M$ to be

$$\int_M \omega := \underbrace{\sum_i}_{\text{finite sum}} \int_{U_i} \rho_i \, \omega \,,$$

each integral is defined in some local parametrization.

where $\{\rho_i\}$ is a partition of unity subordinate to parametrizable open subsets.

The change of variable formula ensures that the integral is well-defined, independent of the choice of parametrizations.

In particular,

**Thm** If $f : N \to M$ is an orientation-preserving diffeomorphism, then $\int_M \omega = \int_N f^* \omega$ for any compactly supported, smooth $m$-form $\omega$.

($m = \dim M = \dim N$)

That is, integral transforms naturally!

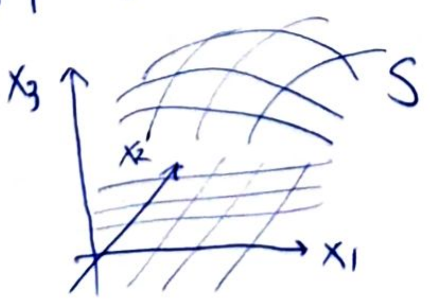
Just like we can integrate an $m$-form over $M$, we can integrate any $p$-form $\omega$ over an oriented $p$-dim'l submanifold $P \overset{i}{\hookrightarrow} M$ by <u>restricting</u> $\omega$ to $P$: $\int_P \omega := \int_P i^* \omega$.

<u>Ex</u> Suppose $\omega = f_1 dx_1 + f_2 dx_2 + f_3 dx_3$ is a smooth 1-form on $\mathbb{R}^3$.
If $\gamma : I \to \mathbb{R}^3$ is a diffeomorphism of $I = [0,1]$ onto $C = \gamma(I)$,

then
$$\int_C \omega = \int_I \gamma^* \omega = \int_0^1 \sum_{i=1}^3 f_i(\gamma(t))\, d\gamma_i$$
$$= \int_0^1 \sum_{i=1}^3 f_i(\gamma(t)) \frac{d\gamma_i}{dt}(t)\, dt .$$

<u>Ex</u> Suppose $\omega = f_1\, dx_2 \wedge dx_3 + f_2\, dx_3 \wedge dx_1 + f_3\, dx_1 \wedge dx_2$ is a compactly supported smooth 2-form on $\mathbb{R}^3$.

Let $S \subset \mathbb{R}^3$ be a surface, which we assume for simplicity to be the graph of a function $G : \mathbb{R}^2 \to \mathbb{R}$ (i.e. $S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 = G(x_1, x_2)\}$)



We can then use the parametrization $h : \mathbb{R}^2 \to S$

$$(x_1, x_2) \mapsto (x_1, x_2, G(x_1, x_2)).$$

Then,

$$\int_S \omega = \int_{\mathbb{R}^2} h^* \omega$$

$$= \int_{\mathbb{R}^2} (f_1 \circ h)\, dx_2 \wedge dG + (f_2 \circ h)\, dG \wedge dx_1 + (f_3 \circ h)\, dx_1 \wedge dx_2$$

$$= \int_{\mathbb{R}^2} \left( (f_1 \circ h)\left(-\frac{\partial G}{\partial x_1}\right) + (f_2 \circ h)\left(-\frac{\partial G}{\partial x_2}\right) + (f_3 \circ h) \right) dx_1 \wedge dx_2$$

Note, $\vec{n} := \left(-\frac{\partial G}{\partial x_1}, -\frac{\partial G}{\partial x_2}, 1\right)$ is normal to $T_x S = \operatorname{Span}\left\{\left(1, 0, \frac{\partial G}{\partial x_1}\right), \left(0, 1, \frac{\partial G}{\partial x_2}\right)\right\}$

at every point $x \in S$.

It follows that we can rewrite the integral as

$$\int_S \omega = \int_{\mathbb{R}^2} (\vec{F} \cdot \vec{u})\, dA$$

where $\vec{F} = (f_1, f_2, f_3)$, $\vec{u} = \frac{\vec{n}}{|\vec{n}|}$, and $dA = |\vec{n}|\, dx_1 \wedge dx_2$, a form familiar from vector calculus.